CROMER, D.	Т.	& 1	IBERMAN	, D). (1970). J	. Chem	. Phy	s. 53,	
1891.										
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DELAPLANE, R. G. & IBERS, J. A. (1969). Acta Cryst. B25, 2423.

DEMEULENAER, J. & TOMPA, H. (1965). Acta Cryst. 19, 1014.

HODGSON, D. J. & IBERS, J. A. (1969). Acta Cryst. B25, 469. HENDRICKS, S. B. (1935). Z. Kristallogr. 91, 48. IBERS, J. A. (1965). Ann. Rev. Phys. Chem. 16, 375.

- International Tables for X-ray Crystallography (1962). Vol. III, pp. 202–209. Birmingham: Kynoch Press.
- LARSON, A. C. (1967). Acta Cryst. 23, 664.
- LINDBLOM, L., MARSH, R. E. & WASER, J. (1972). Acta Cryst. B28, 000.
- PEDERSEN, B. F. (1968). Acta Chem. Scand. 22, 2953.
- STEWART, R. F., DAVIDSON, E. R. & SIMPSON, W. T. (1965). J. Chem. Phys. 42, 3175.

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Variances and Covariances of the Anisotropic Temperature Parameters*

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It is proposed that, in reporting results of crystal structure refinements that include anisotropic temperature parameters, the covariances involving the temperature parameters be expressed in terms of the following six parameters: the lengths of the three principal axes of the vibrational ellipsoids and the orientational angles about these three axes. Equations are presented for deriving the covariances that relate to these new parameters from the covariances given in terms of the primary parameters B_{kl} .

In reporting the results of a crystal structure analysis that includes adjustments of anisotropic thermal parameters, it is the common practice to list the magnitudes U_{ii} (i=1,2,3) of the principal axes of a vibrational ellipsoid and the direction cosines of these axes relative to a crystal coordinate system with axes \mathbf{a}_i . Thus, twelve quantities – three U_{ii} 's and nine direction cosines - are presented, of which only six are independent, because of the constraints on the cosines. Along with these twelve derived parameters are often presented their associated standard deviations (and sometimes covariances as well). The three standard deviations $\sigma(U_{ii})$ of the magnitudes of the principal axes are readily interpretable. It is more difficult, however, to understand the meaning of the nine standard deviations associated with the direction consines, not only because of the constraints among them but also because of the difficulty of visualization. We suggest here an alternative approach to presenting the uncertainties in the orientations of the principal axes. In particular, we suggest that these uncertainties be represented as uncertainties in the angular displacements α_i ($\alpha_1 \equiv \alpha$, $\alpha_2 \equiv \beta$, $\alpha_3 \equiv \gamma$) of the ellipsoid around its principal axes. For example, the quantity $\sigma(y)$ in Fig. 1 represents the uncertainty of orientation of the ellipsoid with respect to a rotation about the eigenvector c. Note that while $\alpha = \beta = \gamma = 0$, the variances and covariances relating to these angles need not be zero.

To derive expressions for variances and covariances

that involve α , β , and γ , we presume that the temperature parameters B_{kl} have been transformed to the coordinate system in which the coordinate axes are parallel to the eigenvectors of the B_{kl} . This is achieved by a similarity transformation that diagonalizes B_{kl} (see, for example, Rollett & Davies, 1955; Waser, 1955; Busing & Levy, 1958) by means of a matrix G_{il} ,

$$U_{ij} = \sum_{k} \sum_{l} G_{ik} G_{jl} B_{kl} = V_i \delta_{ij} .$$
 (1)

The δ_{ij} are Kronecker deltas, and the U_{ij} are the diagonalized values of the temperature parameters; we have found it convenient to introduce the new symbols, $V_i = U_{ii}$. Variances and covariances are attached to all parameters U_{ij} including those that are zero. The covariance matrix associated with the U_{ij} is related to that associated with the B_{kl} by

$$\operatorname{cov} (U_{ij}, U_{kl}) = \sum_{a,b,c,d} \frac{\partial U_{ij}}{\partial B_{ab}} \frac{\partial U_{kl}}{\partial B_{cd}} \operatorname{cov} (B_{ab}, B_{cd})$$
$$= \sum_{a,b,c,d} G_{ia} G_{jb} G_{kc} G_{ld} \operatorname{cov} (B_{ab}, B_{cd}).$$
(2)

† Important aspects of these matters, including the transformation to main axes in oblique crystal systems, were recently summarized by Cerrini (1971). The matrices G_{ik} in (1) depend, of course, on whether covariant or contravariant (or mixed) tensor components are chosen; but our results do not depend on the details of the G_{ik} , and we also do not use index positions to distinguish between covariant, contravariant, or mixed components.

^{*} Contribution No. 4380.

To find the covariances between the new parameters α_i and α_j , and those between α_i and V_j , we must transform cov (U_{ij}, U_{kl}) to the new parameters α_l and V_j , by equations that are analogous to the first portion of (2). Among the quantities needed are the partial derivatives of the V_j with respect to the U_{kl} , which are equal to unity when the three indices are equal, and zero when any two of them are different,

$$\partial V_j / \partial U_{kl} = \delta_{jk} \delta_{jl}$$
 (3)

Hence, the variances and covariances for the parameters V_i are equal to the analogous quantities that pertain to the diagonal components U_{il} ,

$$\operatorname{cov}\left(V_{i}, V_{j}\right) = \operatorname{cov}\left(U_{ii}, U_{jj}\right).$$
(4)

To obtain the covariances that involve the parameters α_k we proceed indirectly, by deriving expressions for the derivatives $\partial U_{ij}/\partial \alpha_k$ rather than $\partial \alpha_k/\partial U_{ij}$. Consider the operator $\mathbf{R}_i(\alpha_i)$ which rotates about the principal axis a_i of the ellipsoid $(a_1 \equiv a, a_2 \equiv b, a_3 \equiv c)$, by an angle α_i $(\alpha_1 \equiv \alpha, \alpha_2 \equiv \beta, \alpha_3 \equiv \gamma)$. The derivatives of the U_{ij} are defined by the expression

$$\frac{\partial \mathbf{U}}{\partial \alpha_k} = \lim_{\alpha_k \to 0} \frac{1}{\alpha_k} \left[\mathbf{R}_k(\alpha_k) \mathbf{U} \mathbf{R}_k^{-1}(\alpha_k) - \mathbf{U} \right].$$
(5)

For example, consider the partial derivatives of the U_{ij} with respect to α , a rotation around the *a* axis:

$$\begin{pmatrix} \frac{\partial U_{ij}}{\partial \alpha} \end{pmatrix} = \lim_{\alpha \to 0} \frac{1}{\alpha} \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix} \\ \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix} - \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix} \end{bmatrix} \\ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (V_2 - V_3) \\ 0 & (V_2 - V_3) & 0 \end{pmatrix}.$$
 (6)

The equations that involve β and γ are analogous and can be written down by suitable cyclic changes in (6).

The substance of our developments can also be obtained by applying first order perturbation methods, as used e.g. in quantum mechanics (see e.g. Pauling & Wilson, 1935; Franklin, 1968), to the eigenvalues and eigenvectors of the temperature parameter ellipsoid. If we let the U_{ij} be varied by ϵ_{ij} ($=\epsilon_{ji}$), the change of the eigenvalues $U_{ii} = V_i$ turns out to be $\delta V_i = \epsilon_{ii}$. Moreover, an expansion of the change $\delta \mathbf{x}^{(i)}$ of the (normalized) eigenvector $\mathbf{x}^{(i)}$ in terms of the original eigenvectors, $\delta \mathbf{x}^{(i)} = \sum e_{ik} \mathbf{x}^{(k)}$, with $e_{kk} = 0$, leads to $e_{ik} = \epsilon_{ik}/(V_i - V_k)$. This result is equivalent to the statement that e.g. $\mathbf{x}^{(1)}$ is rotated about $\mathbf{x}^{(2)}$ by $\delta \beta = \epsilon_{12}/(V_1 - V_2)$ and about $\mathbf{x}^{(3)}$ by $\delta \gamma = -\epsilon_{13}/(V_1 - V_3)$.

We now use the relationships (6) to express co-

variances that pertain to the U_{ij} in terms of covariances that involve the α_k and then invert the resulting equations. For example, we find that

$$\operatorname{cov} (U_{23}, U_{12}) = \sum_{i,k} \frac{\partial U_{23}}{\partial \alpha_i} \frac{\partial U_{12}}{\partial \alpha_k} \operatorname{cov} (\alpha_i, \alpha_k)$$
$$= (V_2 - V_3) (V_1 - V_2) \operatorname{cov} (\alpha, \gamma) \qquad (7)$$
hat

so that

$$\operatorname{cov}(\alpha, \gamma) = \operatorname{cov}(U_{23}, U_{12})/(V_2 - V_3)(V_1 - V_2).$$

In this way we bypass the singular aspects of (6) and of the analogous expressions for β and γ (their Jacobians vanish). The results are

$$\begin{array}{l} \cos\left(V_{1},\alpha\right) = \cos\left(U_{11},U_{23}\right)/(V_{2}-V_{3})\\ \cos\left(V_{1},\beta\right) = \cos\left(U_{11},U_{13}\right)/(V_{3}-V_{1})\\ \cos\left(V_{1},\gamma\right) = \cos\left(U_{11},U_{12}\right)/(V_{1}-V_{2})\\ \cos\left(V_{2},\alpha\right) = \cos\left(U_{22},U_{23}\right)/(V_{2}-V_{3})\\ \cos\left(V_{2},\beta\right) = \cos\left(U_{22},U_{12}\right)/(V_{1}-V_{2})\\ \cos\left(V_{2},\gamma\right) = \cos\left(U_{23},U_{23}\right)/(V_{2}-V_{3})\\ \cos\left(V_{3},\alpha\right) = \cos\left(U_{33},U_{23}\right)/(V_{2}-V_{3})\\ \cos\left(V_{3},\beta\right) = \cos\left(U_{33},U_{13}\right)/(V_{3}-V_{1})\\ \cos\left(V_{3},\gamma\right) = \cos\left(U_{23},U_{12}\right)/(V_{1}-V_{2})\\ \cos\left(\alpha,\alpha\right) = \cos\left(U_{23},U_{13}\right)/(V_{2}-V_{3})^{2}\\ \cos\left(\alpha,\beta\right) = \cos\left(U_{23},U_{13}\right)/(V_{2}-V_{3})\left(V_{3}-V_{1}\right)\\ \cos\left(\alpha,\gamma\right) = \cos\left(U_{23},U_{13}\right)/(V_{2}-V_{3})\left(V_{1}-V_{2}\right)^{2}\\ \cos\left(\beta,\beta\right) = \cos\left(U_{13},U_{13}\right)/(V_{3}-V_{1})^{2}\\ \cos\left(\beta,\gamma\right) = \cos\left(U_{13},U_{12}\right)/(V_{3}-V_{1})\left(V_{1}-V_{2}\right)^{2}. \end{array}$$

If the units of the V_i are Å⁺², the units of cov (V_i, V_j) are Å⁺⁴, those of cov (V_i, α_j) , Å⁺². radian, and those of cov (α_i, α_j) , (radian)².

The covariances between the V_i and α_i , and the positional coordinates of the atom, x_i , can be derived in a similar manner if desired. Using the covariances between the original temperature parameters B_{ab} and the coordinates x_k as starting point, we first go to the U_{ij} by a relationship analogous to (2),



Fig. 1. The standard deviation of γ in relationship to the eigenvectors of the temperature ellipsoid.

$$\operatorname{cov}\left(U_{ij}, x_{k}\right) = \sum_{ab} G_{ia} G_{jb} \operatorname{cov}\left(B_{ab}, x_{k}\right)$$
(8)

and thence

$$\operatorname{cov}(V_i, x_k) = \operatorname{cov}(U_{ii}, x_k)$$
.

To obtain the covariances between α , β , γ and x_k we employ transformations similar to (7), which give

$$\begin{aligned} &\cos(\alpha, x_k) = \cos((U_{23}, x_k)/(V_2 - V_3)) \\ &\cos(\beta, x_k) = \cos((U_{31}, x_k)/(V_3 - V_1)) \\ &\cos(\gamma, x_k) = \cos((U_{12}, x_k)/(V_1 - V_2)). \end{aligned}$$

In summary, we have derived the transformations that yield covariances of the temperature parameters in terms of the quantities V_i and α_j . This system of variables is preferable to two others: (1) to that of the V_i and the directional cosines of the eigenvectors of the temperature parameters, because the directional cosines are not independent of each other, offsetting the favorable aspect of their geometrical significance; (2) to the system of the U_{ij} , which are independent of each other (when $i \leq j$), but where the off-diagonal parameters have no intuitively obvious interpretation. In contrast, the $V_i \equiv U_{ii}$ and α_j are both independent of each other and subject to easy geometric interpretation.*

We note that the advantage of using the V_i and α_j is one of visualization, and not necessarily mathematical. Either of the covariance matrices, that for the U_{ij} (with $i \le j$) and that for the V_i and α_j , contain all available covariance information regarding temperature parameters.

An example

As a numerical example, we choose a carboxylate oxygen atom in the structure of L-N-acetylhistidine monohydrate (Kistenmacher & Marsh, 1971). This compound forms triclinic crystals, space group P1, with two molecules in the unit cell. The data were collected using Cu K α radiation out to $2\theta = 154^{\circ}$; least-squares refinement led to an R index of 0.029 for 2152 reflections. The standard deviations in the positions of the nonhydrogen atoms are about 0.003 Å. The atom we selected has moderately anisotropic thermal motion; the amplitudes of vibration are approximately equal along the minor and intermediate principal axes of motion, but considerably larger in the direction of the major axis. We report all intermediate results for the purposes of checking computer programs, and for the same reasons include more figures than we necessarily believe to be significant.

Cell dimensions:*	a = 8.865 (2) Å	$\alpha = 102.24 (1)^{\circ}$
	b = 9.097(2)	$\beta = 90.30(1)$
	c = 7.346(2)	$\gamma = 117.73(1)$

Anisotropic temperature parameters, in the form $\exp(-b_{11}h^2...-b_{23}kl)$:

$b_{11} =$	8 ·9 68×	10-3
$b_{22} =$	6.945	
$b_{33} =$	24.172	
$b_{12} =$	7.071	
$b_{13} = -$	- 8·135	
<i>b</i> ₂₃ =	2.435	

The variance-covariance matrix for these parameters is shown at the top of Table 1.

To transform the coefficients b_{ij} to main axes it is convenient to write the exponent of the temperature

* The standard deviations of the cell dimensions were not used in the present calculations. The angles α , β , γ are, of course, not related to the angles used to define the orientational variances and covariances of the eigenvectors of the temperature ellipsoid.

Tal	ble	1.	Covariance	matrices	of	temperature	coefficients
	~ ~ ~				~,		000,,00000000

		10 ¹⁰	× cov (b _i	j, b _k ę)		
	b ₁₁	b ₂₂	b ₃₃	b ₁₂	b ₁₃	b ₂₃
b ₁₁	430.5	-7.5	-199.7	351.3	-7.3	-55.2
b ₂₂	-7.5	377.2	-166.7	341.9	70.8	121.1
b ₃₃	-199.7	-166.7	1314.5	-191.1	-124.1	230, 4
b ₁₂	351.3	341.9	-191.1	1107.4	149.4	-12.7
b ₁₃	-7.3	70.8	-124.1	149.4	1804.6	696.7
b ₂₃	-55.2	121.1	230.4	-12.7	696.7	1711.3
		10 ⁸ :	× cov (U _{ii}	(ويال		
	U ₁₁	U22	U ₃₃	U ₁₂	U ₁₃	U23
U.,	36.08	-10.25	-14.71	-2.20	2.18	0.69
U,,	-10.25	45.03	-17,90	-2,89	0.29	-2,50
U ₃₃	-14,71	-17.90	100.97	0.43	1.43	-4.08
U ₁₂	-2,20	-2.89	0.43	23.75	-1.15	1,80
U13	2,18	0.29	1.43	-1.15	36.78	-3.20
U ₂₃	0.69	-2.50	-4.08	1.80	-3.20	38.98
		10		(V: 0°)		
				· · [, ·· [/		
	۷,	V ₂	٧,	α_1	α,	۵,
V 1	0.00361	-0.00103	-0.00147	-0.00141	0.00435	0.1507
V_2	-0.00103	0.00450	-0.00179	0.00513	0.00058	0,1986
ν,	-0.00147	-0.00179	0.01010	0.00837	0.00286	-0.0297
α_1	-0.00141	0.00513	0.00837	1.639	0.131	2.536
С <u>2</u>	0.00435	0.00058	0.00286	0.131	1.458	1,569
α,	0.1507	0.1986	-0.0297	2.536	1.569	1119

^{*} When the variances (i.e. the diagonal terms of the covariance matrix) of the variables V_i and α_j are dominant, the situation regarding the uncertainties of the temperature parameters is particularly easy to visualize, because essentially all information can be conveyed by the standard deviations of the V_i and α_j . The more common case is, however, that there are strong correlations between the errors in at least some of the V_i and α_j . The proper assessment of these errors requires not only a knowledge of the standard deviations but of the (off-diagonal) covariances as well. To report only standard deviations in this case would be an oversimplification that could easily mislead and must be guarded against. The problem of variances and covariances has been considered recently by Johnson (1971). He shows for idealized data that at least approximately the covariance matrix of the temperature parameters, $cov (B_{ij}, B_{kl})$, is related to the positional covariance matrix $cov(x_i, x_j)$.

factor in the form $-(\frac{1}{4}) \sum_{ij} B_{ij}h_ih_j(\mathbf{a}_i^* \cdot \mathbf{a}_j^*)$, where a tively, Å⁴, Å².radian, and radian². Considering just the square roots of the diagonal terms of this matrix we

factor $\frac{1}{2}$ must not be overlooked when going from the b_{ij} to the B_{ij} when $i \neq j$. Similarly, when i = j, k = l, each element of the covariance matrix $cov (B_{ij}, B_{kl})$ follows directly from the related element of $cov(b_{ij}, b_{kl})$, but when i = j and $k \neq l$, or when $i \neq j$ and k = l an additional factor $\frac{1}{2}$ must be used, and a factor $\frac{1}{4}$ when $i \neq j$ and $k \neq l$. In addition to transforming to main axes, we divide by $8\pi^2$, which yields the diagonal matrix U_{ij} . The values of the U_{ii} and of the directional cosines of these eigenvectors relative to the crystallographic axes are: L

		а	D	С
$U_{11} = 0.02139 \text{ Å}^2$	U_{11}	-0.0586	0.8801	0.0204
$U_{22} = 0.02285$	$U_{22}^{}$	0.9359	-0.4692	0.3441
$U_{33} = 0.07161$	U_{33}	0.3474	0.0730	-0.9387

Note that these eigenvectors should form a righthanded system of axes for a proper connection with the sense of rotation around these eigenvectors.

The covariance matrix of the U_{ij} follows from that of the B_{ij} by equations (1) and (2). The result is given in the middle of Table 1. The final covariances between the V_i and V_j , the V_i and α_j , and the α_i and α_j , are shown in the bottom matrix of Table 1. The units for the different terms just enumerated are, respecsquare roots of the diagonal terms of this matrix we find

$V_1 = 0.0214 \pm 0.0006 \text{ Å}^2$	$\sigma(\alpha_1) = \pm 0.7^\circ$
$V_2 = 0.0228 \pm 0.0007$	$\sigma(\alpha_2) = \pm 0.7$
$V_3 = 0.0716 \pm 0.0010$	$\sigma(\alpha_3) = \pm 19.$

Of special note is the relatively large value of the standard deviation associated with α_3 , which reflects the near equality of V_1 and V_2 . We emphasize, however, that these standard deviations alone do not give the complete story, because the covariances are by no means negligible.

References

BUSING, W. R. & LEVY, H. A. (1958). Acta Cryst. 11, 450. CERRINI, S. (1971). Acta Cryst. A 27, 130.

FRANKLIN, J. N. (1968). Matrix Theory. Englewood Cliffs, N. J.: Prentice-Hall.

JOHNSON, C. K. (1971). Abstract B3 of the Summer Meeting of the American Crystallographic Association, August 15-20, 1971, Ames, Iowa.

- KISTENMACHER, T. J. & MARSH, R. E. (1971). Science, 172, 945.
- PAULING, L. & WILSON, E. B. (1935). Introduction to Quantum Mechanics, New York: McGraw-Hill.
- ROLLETT, J. S. & DAVIES, D. R. (1955). Acta Cryst. 8, 125. WASER, J. (1955). Acta Cryst. 8, 731.

Acta Cryst. (1972). B28, 2201

The Crystal Structure of Benzamide

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The structure of benzamide has been refined using counter-measured intensity data (1037 planes). Libration corrections have been applied to the full-matrix least-squares refined parameters. The principal interatomic distances are C-C (mean benzene ring) 1.391 (5), C-O 1.249 (3), C-N 1.342 (3) and C-C (carboamide) 1.501(4) Å.

Previous work

Earlier determinations of the crystal structure of benzamide have been reported, by Penfold & White (1959) and Blake & Small (1959). Both of these were based upon two-dimensional photographically recorded data and are consequently, by present day standards, of limited accuracy. The results reported here are of a recent refinement making use of more extended experi-

Experimental

University of Birmingham.

An evenly developed crystal of mean dimension 0.2mm was grown from benzene solution and used for the experimental measurements. The crystal was mounted on the three-circle diffractometer of Small &

mental data collected at the Chemistry Department,